SMALL ELASTOPLASTIC STRAINS OF DIELECTRICS

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The theory of small elastoplastic strains of dielectrics in the presence of an electric field is developed in the work. An expression is obtained for the electric part of the stress tensor. Total free energy minimum theorems with constant temperature and simple proportional loading and unloading theorems are proved. Experiments to calculate the mass functions are discussed.

1. The theory of small elastoplastic strains and its fundamental theorems and corollaries [1] were generalized in [2] to the case of instantaneous small elastoplastic strains. There also arises the problem, to be discussed below, regarding the development of a theory of small elastoplastic strains of dielectrics in the presence of an electric field.

Let us consider a solid dielectric of arbitrary shape with volume V bounded by the surface S. We introduce in a nondeformed coordinate system x_i, the u_i, or components of the displacement vector and ε_{ij}, the strain tensor components. We will henceforth assume that volume forces with cubic density F_i act on the body and stresses T_i on the surface S. The electric field E is created by external charges distributed with density ρ within V and by free charges with density ρ_Σ on the conductor surface Σ. It is necessary to solve the electrostatics problem defined by Maxwell's equations and by the corresponding boundary conditions, taking into account deformation of the dielectric if we are given F_i, T_i, ρ, and ρ_Σ. Suppose the dielectric medium is isotropic in the unstressed state in terms of electrical and mechanical properties. In general, strain destroys the isotropy of the body. Henceforth we will assume that the dielectric is also isotropic following strain

$$\varkappa=\varkappa_0+f(\varepsilon_u,\ \theta),$$

where $\varkappa_0$ is the permittivity of the medium in the undeformed state, $\theta=\varepsilon_{ii}$, and ε_u is strain intensity:

$$\varepsilon_u=\left[\frac{2}{3}e_{ij}e_{ij}\right]^{1/2}. \tag{1.1}$$

Here, e_{ij} is the strain deviator.

Following [3], we obtain an expression for the stress tensor due to the electric field,

$$\sigma_{ij}^e=-\frac{1}{2}E^2\left(\frac{\partial f}{\partial\varepsilon_u}\frac{2}{3}\frac{e_{ij}}{\varepsilon_u}+\frac{\partial f}{\partial\theta}\delta_{ij}\right).$$

The equilibrium equations, taking into account Coulomb forces, are written in the form

$$\frac{\partial\sigma_{ij}}{\partial x_j}+\rho E_i+F_i=0,$$

where $\sigma_{ij}=\sigma_{ij}^0+\sigma_{ij}^e$, and where σ_{ij}^0 is the mechanical part of the stress tensor related to the variation in the mechanical part of free energy with variations in $\delta\varepsilon_{ij}$.

We have, in accordance with [3] for the total stress tensor T_{ij},

$$T_{ij}=\sigma_{ij}^0+\sigma_{ij}^e+\varkappa_0\left(E_iE_j-\frac{E^2}{2}\delta_{ij}\right).$$

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The boundary conditions for the stresses on surfaces with normals having directional cosines n_i will be written in the form

$$T_{ij}n_j = \left(E_{1i}E_{1j} - \frac{E_1^2}{2}\delta_{ij}\right)n_j + T_i \quad x \in S - \Sigma_1; \tag{1.2}$$
$$T_{ij}n_j = T_i \quad x \in \Sigma_1.$$

Here, Σ_i is the part of the conductor surface coinciding with the part of the surface S of the dielectric. We represent the tensor σ_{ij}^0 in a form of a deviator and a spherical tensor

$$\sigma_{ij}^0 = S_{ij}^0 + \sigma^0\delta_{ij}; \quad \sigma^0 = \frac{1}{3}\sigma_{ii}^0.$$

We set

$$\sigma_u = \left[\frac{3}{2}S_{ij}^0 S_{ij}^0\right]^{1/2} \quad (E_iE_i)^{1/2} = E; \; D = (D_iD_i)^{1/2},$$

where the D_i are the components of the induction vector. We will assume that loading is simple or nearly simple [1]. A calculation of simple loading, both mechanical and electric, will be given below.

The theory of small elastoplastic strains of dielectrics with simple active loading is described by three laws. Firstly, the magnitudes σ^0 and ε obey Hooke's law.

$$\sigma^0 = 3k\varepsilon. \tag{1.3}$$

Secondly, the deviator values are related by the equation

$$S_{ij}^0 = \frac{2}{3}\frac{\sigma_u}{\varepsilon_u}e_{ij}. \tag{1.4}$$

Finally,

$$E_i = \frac{E}{D}D_i. \tag{1.5}$$

We further add to Eqs. (1.3)-(1.5) three equations defining three universal functions that satisfy the correspondence principle when no field is present:

$$k = \varphi_1(D); \quad \sigma_u = \varphi_2(\varepsilon_u, D);$$
$$E = \varphi_3(\theta, \varepsilon_u)D; \quad \varphi_3(\theta, \varepsilon_u) = 1/\varkappa. \tag{1.6}$$

2. The function φ_1 occurring in Eq. (1.6) can be determined from a uniform compression experiment on a thin, flat, dielectric sphere, φ_2 can be determined from a linear compression experiment on a thin tablet in a homogeneous field, and φ_3 can be determined from a tension and torsion experiment on a thin-walled tube in a varying field.

We will consider the uniform compression problem in the high-pressure chamber of a solid dielectric medium, filling the space between the plates of a spherical capacitor. The pressure p is uniformly distributed with respect to the inner and outer surfaces of the hollow sphere. The distance c between the plates is small: $c \ll R$, where R is the mean radius of the plates. Stress and strain can therefore be considered homogeneous to an approximation. According to Eq. (1.2),

$$\sigma_{rr}^{(0)} - \frac{1}{2}\frac{D^2}{\varkappa^2}\left(\frac{\partial f}{\partial \varepsilon_u}\frac{2}{3}\frac{e_{rr}}{\varepsilon_u} + \frac{\partial f}{\partial \theta}\right) + \frac{D^2}{2\varkappa_0} = -p; \tag{2.1}$$
$$\sigma_{\varphi\varphi}^0 - \frac{1}{2}\frac{D^2}{\varkappa^2}\left(\frac{\partial f}{\partial \varepsilon_u}\frac{2}{3}\frac{e_{\varphi\varphi}}{\varepsilon_u} + \frac{\partial f}{\partial \theta}\right) - \frac{D^2}{2\varkappa_0} = 0;$$
$$\sigma_{\theta\theta}^0 - \frac{1}{2}\frac{D^2}{\varkappa^2}\left(\frac{\partial f}{\partial \varepsilon_u}\frac{2}{3}\frac{e_{\theta\theta}}{\varepsilon_u} + \frac{\partial f}{\partial \theta}\right) - \frac{D^2}{2\varkappa_0} = 0.$$

Adding Eqs. (2.1) and noting that $e_{ii} = 0$, we obtain

$$3\sigma^{(0)} = -p + \frac{3}{2}\frac{D^2}{\varkappa_0^2}\frac{\partial f}{\partial \theta} + \frac{D^2}{2\varkappa_0}; \quad \sigma^{(0)} = \varphi_1(D)\,\theta. \tag{2.2}$$

It is evident from Eq. (2.2) that if we know $\varkappa(\theta, \varepsilon_u)$ and measure p, θ, D, and $\varkappa$, we can obtain $\varphi_1(D)$. The form of the dependence of $\varkappa(\theta, \varepsilon_u)$ on θ and ε_u will be determined below.

Let us consider the strain problem for a dielectric plate within a plane capacitor and filling it. The axis z is directed along the normal to the capacitor plates. We assume that the plate dimensions (equal to a in the direction of the x axis and to b in the direction of the y axis) satisfy the condition $a \gg b$, but at the same time a, $b \gg d$, where d is plate thickness. In this case, the plate can be approximately considered as a bar. Suppose a stress T_x acts in the direction of the x axis and let all components of the strain tensor, except for $\varepsilon_{xx} = \varepsilon_1$, be set equal to 0. It is necessary to restrict strain along the y axis by means of undeformable walls in order to realize linear compression in the direction of the x axis. It can be verified that if we set ε_{xx}=const, the boundary conditions and equilibrium equations are satisfied. Thus the plate experiences homogeneous strain. Then

$$e_{xx}=2/3\varepsilon_1,\ e_{yy}=-\ \varepsilon_1/3;\ e_{zz}=-\ \varepsilon_1/3;$$
$$e_{xy}=e_{xz}=e_{yz}=0;\ \theta=\varepsilon_1,\ \varepsilon_u=2/3\varepsilon_1.$$

We have, in accordance with Eq. (1.2), on the end faces of the plate

$$\frac{2}{3}\sigma_u-\frac{1}{3}\frac{D^2}{\varkappa^2}\frac{\partial f}{\partial \varepsilon_u}-\frac{1}{2}\frac{D^2}{\varkappa^2}\frac{\partial f}{\partial \theta}-k\varepsilon_1=\frac{D^2}{2}+\frac{D^2}{2\varkappa_0}+T_x.$$

We can find $\varphi_2(\varepsilon_u, D)$ by measuring ε_1, T_x, $\varkappa$, and D if we know $\varphi_1(D)$ and $\varphi_3(\theta, \varepsilon_u)$. We have by the tension and torsion experiment for a thin-walled dielectric tube

$$\sigma^0_{xy}=\tau;\quad \sigma^0_{xx}=\sigma^0_{yy}=\sigma^0_{xz}=\sigma^0_{yz}=0;\ \sigma^0{}_{zz}=\sigma_1; \tag{2.3}$$
$$\varepsilon_{xy}=\gamma;\ \varepsilon_{xx}=\varepsilon_{yy}=-1/m\varepsilon_{zz},\ \varepsilon_{zz}=e_1. \tag{2.4}$$

The curve $m=m(e_i, \gamma)$ was also found in the experiment. It can be easily verified that Eqs. (2.3) and (2.4) satisfy the equilibrium equations and boundary conditions. We obtain from Eqs. (1.1), (2.3), and (2.4),

$$\varepsilon_u=\frac{\sqrt{2}}{3}\sqrt{\frac{2(m-1)^2}{m^2}e_1^2+\frac{3}{2}\gamma^2};\qquad \theta=\left(\frac{m-2}{m}\right)e_1.$$

We obtain the function $\varphi_3(\theta, \varepsilon_u)$ of θ and ε_u, which is the coefficient in the third equation of Eq. (1.6), by measuring the capacitance and dimensions of the coaxial cable corresponding to a thin-walled tube in a weak varying field (when striction can be neglected).

3. Free energy Ψ can be represented by the space integral over the entire space

$$\Psi=\int\left[\int_0^M(\sigma_{ij}\delta\varepsilon_{ij}+E_k\delta D_k)\right]dV, \tag{3.1}$$

where 0 is the state characterizing the absence of an electric field and strain and M is the state in the presence of an electric field and strains. We assume that the integrand in Eq. (3.1) has the form of a total differential.

We calculate the continuous virtual displacement δu_i and induction δ D to formulate the free-energy minimum theorem by means of the equations

$$\begin{aligned}&\delta u_i=u_i'-u_i, && x\in V;\\ &\delta u_i=0, && x\in S;\\ &dw\,\delta\mathrm{D}=0, && \oint_\Sigma \delta\mathrm{D}\,d\Sigma=0.\end{aligned}$$

Here it is assumed that

$$F_i=0,\quad \rho=0.$$

The free-energy minimum theorem asserts that the true equilibrium state of a body whose material has been strengthened differs from the virtual viscous state in that the total free energy at constant temperature is at a minimum for the true state. To prove the theorem, we use the condition

$$\sigma_u\geqslant\frac{1}{2}E^2\frac{\partial f}{\partial\varepsilon_u}$$

and the strengthening condition

$$\frac{\partial S_u}{\partial\varepsilon_u}>0;\ \ \frac{\partial S_u}{\partial\varepsilon_u}\frac{\partial\sigma}{\partial\varepsilon_{\alpha\alpha}}>\left(\frac{\partial S_u}{\partial\varepsilon_{\alpha\alpha}}\right)^2;I=\frac{D(S_u,\sigma,E)}{D(\varepsilon_u,\varepsilon_{\alpha\alpha},D)}>0.$$

A simple proportional loading theorem for dielectrics in an electric field can be proved for materials defined by the equations

$$\varepsilon_{\alpha\alpha}=0; \quad k=\infty; \quad S_u=\frac{\partial\Phi}{\partial\varepsilon_u}; \quad E=\frac{\partial\Phi}{\partial D}; \tag{3.2}$$
$$\Phi = A\varepsilon_u^{\gamma+1} + D^2B(\varepsilon_u); \; A, \; \gamma=\text{const.}$$

Let us say that loading is simple if the directional tensors of the mechanical stresses [1] and the unit field strength vectors are constant in time. Let us say that a process in that stresses, strains, and induction vary in proportion to their time-dependent parameters, is a process of proportional resistance to loading. The variation of mass forces, surface forces, and the electric field in proportion to the time-dependent parameters, is called proportional loading.

It was shown that the simple proportional loading of material given by Eq. (3.2) in the presence of an electric field is a process whose resistance is not proportional to loading.

It can be shown that the elastic unloading problem for a dielectric in increments of the corresponding values will not have the form of the elastic dielectric problem, which is explained by the nonlinearity of the electrical terms in the equilibrium equations and boundary conditions with respect to the field. If the unloading problem is solved by the method of successive approximations, it is necessary to separately solve at each state an electrostatics problem and unloading equation for the known strain state found from the preceding approximations, and a mechanical problem for known fields and inductions. The elastic unloading problem with known fields and inductions in increments of the mechanical values will have the form of the elastic problem under the condition that increments of electric forces act on the body.

Therefore, as in the absence of an electric field, we have an unloading theorem true at each state of the successive approximations. Displacements at a given moment of the unloading stage differ from their values at the moment unloading begins by the size of the elastic displacements which arise in the body if external mass forces found in the preceding approximation and equal to the force differences acting on the body at the given moments are applied to it in the natural state. This is also the case with strains, inductions, and stresses. A theorem for the residual stresses, strains, and displacements with the complete removal of external loads and the field is obtained as a corollary of the elastic unloading theorem.

LITERATURE CITED

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